


Prime Ideals and Maximal Ideals

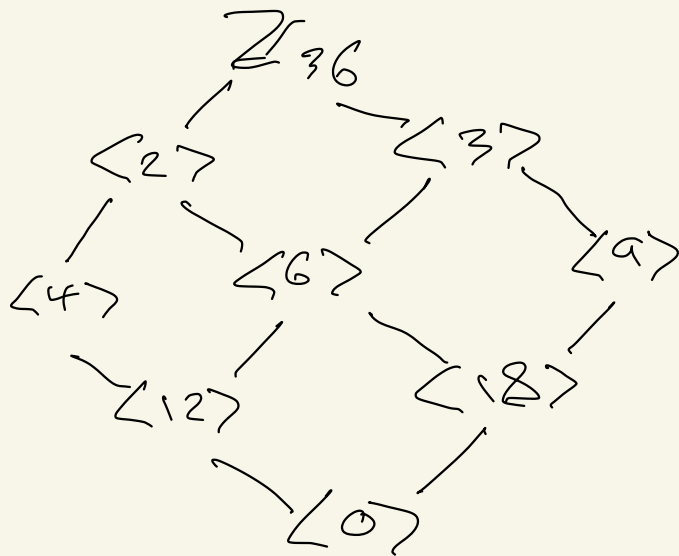
Def An ideal I of a commutative ring R is prime if I is a proper ideal of R and if $ab \in I$ for $a, b \in R$ then $a \in I$ or $b \in I$

Def An ideal I of a commutative ring R is maximal if I is a proper ideal of R and if whenever J is another ideal of R with $I \subseteq J \subseteq R$, then $J = I$ or $J = R$

Ex Let $n \in \mathbb{Z}^+$.

The ideal $n\mathbb{Z}$ of \mathbb{Z}
is prime iff n is prime

Ex This is the lattice
of ideals in \mathbb{Z}_{36}



This shows that $\langle 2 \rangle$ and $\langle 3 \rangle$
are the only maximal ideals of \mathbb{Z}_{36}

Theorem* Let R be
a commutative ring
with unity and let \underline{I}
be an ideal of R . Then
 \underline{I} is prime iff R/\underline{I}
is an integral domain

Proof

R/\underline{I} is an integral domain

iff \underline{I} is proper and R/\underline{I}

has no zero divisors

iff \underline{I} is proper and for any $a, b \in R$

if $(a + \underline{I})(b + \underline{I}) = \underline{I}$ then $a + \underline{I} = \underline{I}$
or $b + \underline{I} = \underline{I}$

iff I is proper and for
any $a, b \in R$ if $ab + I = I$
then $a + I = I$ or $b + I = I$

iff I is proper and for
any $a, b \in R$ if $ab \in I$
then $a \in I$ or $b \in I$

iff I is prime

Theorem* Let R be a commutative ring with unity and let I be an ideal of R . Then I is a maximal ideal iff R/I is a field

Proof

(\Leftarrow) Suppose that R/I is a field and J is an ideal of R that properly contains I . Let $a \in J \setminus I$.

Then $a + I$ is a nonzero element of R/I and therefore since R/I is a field there exists $b \in R$ with $(a + I)(b + I) = 1 + I$

So $ab + \underline{I} = 1 + \underline{I}$.

This implies that $1 = ab + c$
for some $c \in \underline{I}$

So $1 \in \underline{J}$ since $a \in \underline{J}$

and $c \in \underline{I} \subseteq \underline{J}$. This implies
that $\underline{J} = R$

(\Rightarrow) Suppose that \underline{I} is maximal
and let $a \in R \setminus \underline{I}$. We must
show that $a + \underline{I}$ is a unit.

Consider $\underline{J} = \langle a \rangle + \underline{I}$.

This is an ideal that properly
contains \underline{I} .

Since I is maximal,
we must have $J = R$
so $1 \in J$ i.e. $1 = ab + c$
for some $c \in I$. Then

$$\begin{aligned} 1 + I &= ab + c + I = ab + I \\ &= (a + I)(b + I) \end{aligned}$$

so $a + I$ is a unit.

This shows that R/I
is a field

Def Let I and J
be ideals of a ring R

we define

$$I + J = \{a + b \mid a \in I \text{ and } b \in J\}$$

$$IJ = \{a_1 b_1 + a_2 b_2 + \dots + a_n b_n \mid \\ a_i \in I, b_i \in J, n \in \mathbb{Z}^+\}$$

Exercise Prove that
both $I + J$ and IJ
are ideals of R