Prime Ideals and Maximal Ideals

Def An ideal I of a commutative ring R is prime if I is a proper ideal of R and if $ab \in I$ for $a, b \in R$ then $a \in I$ or $b \in I$

Def An ideal I of a commutative ring R is maximal if I is a proper ideal of R and if whenever J is another ideal of R with $I \subseteq J \subseteq R$, then $J = I$ or $J = R$

__Ex__ Let $n \in \mathbb{Z}^+$.
The ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is prime iff n is prime

__~~Ex~~__ This is the lattice of ideals in $\mathbb{Z}_{36}$

$$\begin{tikzcd}
 & & \mathbb{Z}_{36} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \langle 2 \rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle 3 \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
\langle 4 \rangle \arrow[dr, dash] & & \langle 6 \rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle 9 \rangle \arrow[dl, dash] \\
 & \langle 12 \rangle \arrow[dr, dash] & & \langle 18 \rangle \arrow[dl, dash] & \\
 & & \langle 0 \rangle & &
\end{tikzcd}$$

This shows that $\langle 2 \rangle$ and $\langle 3 \rangle$ are the only maximal ideals of $\mathbb{Z}_{36}$

Theorem* Let R be a commutative ring with unity and let I be an ideal of R. Then I is prime iff R/I is an integral domain

Proof

R/I is an integral domain iff I is proper and R/I has no zero divisors iff I is proper and for any $a, b \in R$ if $(a+I)(b+I) = I$ then $a+I = I$ or $b+I = I$

iff I is proper and for any $a,b \in R$ if $ab+I = I$ then $a+I = I$ or $b+I = I$

iff I is proper and for any $a,b \in R$ if $ab \in I$ then $a \in I$ or $b \in I$

iff I is prime

Theorem* Let R be a commutative ring with unity and let I be an ideal of R. Then I is a maximal ideal iff R/I is a field

Proof

($\Leftarrow$) Suppose that R/I is a field and J is an ideal of R that properly contains I. Let $a \in J \setminus I$. Then $a+I$ is a nonzero element of R/I and therefore since R/I is a field there exists $b \in R$ with $(a+I)(b+I) = 1+I$

So $ab + I = 1 + I$.

This implies that $1 = ab + c$ for some $c \in I$

So $1 \in J$ since $a \in J$ and $c \in I \subseteq J$. This implies that $J = R$

($\Rightarrow$) Suppose that I is maximal and let $a \in R \setminus I$. We must show that $a + I$ is a unit.

Consider $J := \langle a \rangle + I$.

This is an ideal that properly contains I.

Since I is maximal, we must have $J = R$ so $1 \in J$ i.e. $1 = ab + c$ for some $c \in I$. Then

$$1 + I = ab + c + I = ab + I$$
$$= (a+I)(b+I)$$

so $a + I$ is a unit.

This shows that R/I is a field

Def Let I and J be ideals of a ring R. We define

$I+J = \{a+b \mid a \in I \text{ and } b \in J\}$

$IJ = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid a_i \in I, b_i \in J, n \in \mathbb{Z}^+\}$

Exercise Prove that both $I+J$ and IJ are ideals of R