1. Let a belong to a ring R. Let $S = \{x \in R \mid ax = 0\}$. Prove that S is a subring of R.
   Solution:
   (i)We have $a0 = 0$. Therefore $0 \in S$ and so $S \neq \emptyset$
   (ii) Assume that $x, y \in S$. Then $ax = 0$ and $ay = 0$. We have
   $a(x - y) = ax - ay = 0 - 0 = 0$ so $x - y \in S$
   (iii) Assume that $x, y \in S$. Then $ax = 0$ and $ay = 0$. We have $a(xy) = (ax)y = 0y = 0$ so $xy \in S$
   From (i), (ii) and (iii) S is a subring of R.

2. Let R be a ring. The center of R is the set $\{x \in R \mid ax = xa \text{ for all } a \in R\}$. Prove that the center of a ring is a subring.
   Solution:
   We will denote the center of R by S.
   (i)We have $0a = a0$ for all $a \in R$. Therefore $0 \in S$ so $S \neq \emptyset$.
   (ii) Assume that $x, y \in S$. Then $ax = xa$ and $ay = ya$ for all $a \in R$. We have
   $a(x - y) = ax - ay = xa - ya = (x - y)a$ for all $a \in R$. Therefore $x - y \in S$.
   (iii) Assume that $x, y \in S$. Then $ax = xa$ and $ay = ya$ for all $a \in R$. We have
   $a(xy) = (ax)y = (xa)y = x(ay) = x(ya) = (xy)a$ for all $a \in R$. Therefore $xy \in S$
   From (i), (ii) and (iii) S is a subring of R.

3. Suppose that a and b belong to a commutative ring R. If a is a unit in R and $b^2 = 0$, prove that $a + b$ is a unit in R.
   Solution:
   Since R is commutative and $b^2 = 0$, therefore
   $(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - ab + ab - b^2 = a^2$
   Since a is a unit, therefore a has a multiplicative inverse a^{-1}. Let $c = (a - b)a^{-2}$. Then from the above it follows that $(a + b)c = 1$.Since R is commutative we also get
   $c(a + b) = 1$. Therefore $a + b$ is a unit.

4. A Boolean ring R is a ring with the property that $a^2 = a$ for all $a \in R$. Prove that any Boolean ring is commutative.
   Solution:
   First we show that for any $a \in R$ we have $a = -a$: Let a be an element of R. Then we have $-a = (-a)^2 = (-a)(-a) = a$ so $a = -a$
   Now let $a, b \in R$. Then we have $a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$. It follows that $ab + ba = 0$. Therefore $ab = -ba$. But $ba = -ba$ by what we showed above. Therefore $ab = ba$. This proves that R is commutative.

5. Consider the set $S = \left\{ \begin{bmatrix} a & c \\ c & b \end{bmatrix} : a, b, c \in \mathbb{Z} \text{ and } c = a - b \right\}$. Prove or disprove that S is a subring of $M_{22}(\mathbb{Z})$.
   Solution:
   (i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ so $S \neq \emptyset$
   (ii)Suppose that $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \in S$ and $B = \begin{bmatrix} a' & c' \\ c' & b' \end{bmatrix} \in S$ so $c = a - b$ and $c' = a' - b'$. Then
   $A - B = \begin{bmatrix} a - a' & c - c' \\ c - c' & b - b' \end{bmatrix}$. We have
   $c - c' = a - b - (a' - b') = a - b - a' + b' = (a - a') - (b - b')$. Therefore $A - B \in S$.

(iii)Suppose that $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \in S$ and $B = \begin{bmatrix} a' & c' \\ c' & b' \end{bmatrix} \in S$ so $c = a - b$ and $c' = a' - b'$. Then $AB = \begin{bmatrix} aa' + cc' & ac' + cb' \\ ca' + bc' & cc' + bb' \end{bmatrix}$. We have
$ac' + cb' = a(a' - b') + (a - b)b' = aa' - ab' + ab' - bb' = aa' - bb'$
$ca' + bc' = (a - b)a' + b(a' - b') = aa' - ba' + ba' - bb' = aa' - bb'$
$(aa' + cc') - (cc' + bb') = aa' + cc' - cc' - bb' = aa' - bb'$
From the above we see that $AB \in S$
From (i), (ii) and (iii) S is a subring of $M_{22}(\mathbb{Z})$

6. Consider the set $S = \{(a, b, c) : a, b, c \in \mathbb{Z} \text{ and } c = a + b\}$. Prove or disprove that S is a subring of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.
Solution:
Consider $x = (1, 0, 1)$ and $y = (0, 1, 1)$. Both x and y belong to S. However $xy = (0, 0, 1)$ which is not an element of S. Therefore S is not a subring of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.