1. Let R be a ring with unity and G a finite nontrivial group. For any $g \in G$ with $g \neq e$, prove that $g - 1$ is a zero divisor in RG.
   Solution:
   Since G is a finite group, therefore $g^n = e$ for some integer $n > 1$. We have $(g-1)(g^{n-1} + g^{n-2} + ... + 1) = g^n - 1 = 0$. Also $(g^{n-1} + g^{n-2} + ... + 1)(g-1) = g^n - 1 = 0$. Note that both $g - 1$ and $g^{n-1} + g^{n-2} + ... + 1$ are nonzero. This shows that $g - 1$ is a zero divisor.

2. Let R be a ring with unity and $G = \{g_1, g_2, ..., g_n\}$ a finite group. Prove that the element $x = g_1 + g_2 + ... + g_n$ is in the center of RG.
   Solution:
   Claim: Let G be a group and $g \in G$. The map $\phi : G \to G$ defined as $\phi(a) = ga$ is a bijection.
   We now prove this claim: Suppose that $a, b \in G$ with $\phi(a) = \phi(b)$. Then $ga = gb$. Multiplying both sides of this equality by g^{-1} on the left gives $a = b$. This shows that ϕ is one-to-one. To show that ϕ is onto, let $a \in G$. Then $\phi(g^{-1}a) = g(g^{-1}a) = (gg^{-1})a = ea = a$. This shows that ϕ is onto. Thus ϕ is a bijection.
   If $g \in G$ the above claim implies that $gx = x$. Similarly one can show that $xg = x$. We now show that x is in the center of RG: Let $\alpha \in RG$. Then $\alpha = r_1g_1 + r_2g_2 + ... + r_ng_n$ for some $r_i \in R$. We have $\alpha x = (r_1g_1 + r_2g_2 + ... + r_ng_n)x = r_1g_1x + r_2g_2x + ... + r_ng_nx = r_1x + r_2x + ... + r_nx = r_1xg_1 + r_2xg_2 + ... + r_nxg_n = xr_1g_1 + xr_2g_2 + ... + xr_ng_n = x(r_1g_1 + r_2g_2 + ... + r_ng_n) = x\alpha$.
   This shows that x is in the center of RG

3. Find all the units and zero divisors of $\mathbb{Z}_6 \times \mathbb{Z}_4$.
   Solution:
   Claim: Let R and S be rings with unity. Then the units of $R \times S$ are $\{(a, b) \mid a \text{ is a unit of } R \text{ and } b \text{ is a unit of } S\}$.
   We now prove this claim. Assume that (a, b) is a unit of $R \times S$. Then there exists $(a', b') \in R \times S$ and $(a'', b'') \in R \times S$ such that $(a', b')(a, b) = (1, 1)$ and $(a, b)(a'', b'') = (1, 1)$. This implies that $a'a = 1$, $b'b = 1$, $aa'' = 1$ and $bb'' = 1$. Therefore a is a unit of R and b is a unit of S. Conversely if a is a unit of R and b is a unit of S then there exist $a', a'' \in R$ and $b', b'' \in S$ such that $a'a = 1$, $aa'' = 1$, $b'b = 1$ and $bb'' = 1$. Then we have $(a', b')(a, b) = (1, 1)$ and $(a, b)(a'', b'') = (1, 1)$ so (a, b) is a unit of $R \times S$.
   The units of $\mathbb{Z}_6$ are 1 and 5 and the units of $\mathbb{Z}_4$ are 1 and 3 so from the above claim the units of $\mathbb{Z}_6 \times \mathbb{Z}_4$ are $\{(1, 1), (1, 3), (5, 1), (5, 3)\}$.
   Claim: Let $R \neq \{0\}$ and $S \neq \{0\}$ be commutative rings. Define $A = \{(r, s) \mid (r, s) \in R \times S \setminus \{(0, 0)\} \text{ and } r = 0 \text{ or } s = 0\}$ and $B = \{(r, s) \mid (r, s) \in R \times S, r \neq 0, s \neq 0 \text{ and } r \text{ is a zero divisor or } s \text{ is a zero divisor}\}$. The zero divisors of $R \times S$ are $A \cup B$.
   We now prove the claim. Assume that $(r, s) \in A$. Then $(r, s) \neq (0, 0)$ and $r = 0$ or $s = 0$. If $r \neq 0$ and $s = 0$ then choose $s' \in S \setminus \{0\}$. We have $(r, 0)(0, s') = (0, 0)$. If $s \neq 0$ and $r = 0$ then choose $r' \in R \setminus \{0\}$. We have $(0, s)(r', 0) = (0, 0)$. This shows that all elements of A are zero divisors. Now let $(r, s) \in B$. Then $r \neq 0$, $s \neq 0$ and either r is a zero divisor or s is a zero divisor. Assume that r is a zero divisor. Then there exists $r' \in R \setminus \{0\}$ with $rr' = 0$. Then we have $(r, s)(r', 0) = (0, 0)$. Since $R \times S$ is commutative this shows that (r, s) is a zero divisor of $R \times S$. Similarly if s is a zero divisor then (r, s) is a zero divisor of $R \times S$. Thus we have shown that elements of B are zero divisors of $R \times S$. To complete the proof of the claim we need to show that if (r, s) is a zero divisor of $R \times S$, then $(r, s) \in A \cup B$. Assume that (r, s) is a zero divisor. If $r = 0$ or $s = 0$ then $(r, s) \in A$.

Otherwise $r \neq 0$ and $s \neq 0$. Since (r, s) is a zero divisor there exists $(r', s') \in R \times S \setminus \{(0, 0)\}$ with $(r, s)(r', s') = (0, 0)$. so $rr' = 0$ and $ss' = 0$. Since $(r', s') \neq (0, 0)$ therefore $r' \neq 0$ or $s' \neq 0$. Since both R and S are commutative, this implies that r is a zero divisor or s is a zero divisor so $(r, s) \in B$. This completes the proof of the claim.

The only zero divisor of $\mathbb{Z}_4$ is 2 and the zero divisors of $\mathbb{Z}_6$ are $2, 3$ and 4. From the above claim the zero divisors of $\mathbb{Z}_6 \times \mathbb{Z}_4$ are
$\{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (0, 1), (0, 2), (0, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (1, 2), (5, 2)\}$

4. Find the characteristics of the following rings:
   (a) $\mathbb{Z}_4 \times \mathbb{Z}_6$ (b) $\mathbb{Z}_4 \times 4\mathbb{Z}$ (c) $M_{22}(\mathbb{Z}_5)$
   (d) The group ring RG where $R = \mathbb{Z}_4$ and $G = \mathbb{Z}_8$
   Solution:
   (a) lcm(4,6)=12 (b) 0 (c) 5 (d) 4

5. An element a of a ring is an idempotent if $a^2 = a$. Prove that the only idempotents of an integral domain are 0 and 1.
   Solution:
   Note that 0 and 1 are certainly idempotents. For the other containment, assume that a is an idempotent of an integral domain. Then $a^2 = a$ so $a^2 - a = 0$. This implies that $a(a - 1) = 0$. Since an integral domain has no zero divisors this implies that $a = 0$ or $a - 1 = 0$ i.e. $a = 0$ or $a = 1$.

6. In a commutative ring of characteristic 2, prove that the idempotents forms a subring.
   Solution:
   Let R be the ring in question and let S be the set of its idempotents. Note that since $\text{char}(R) = 2$ we have $a = -a$ for any a in R
   (i)$0^2 = 0$ so $0 \in S$. Therefore $S \neq \emptyset$
   (ii) Let $a, b \in S$. Then $a^2 = a$ and $b^2 = b$. Since R is commutative and $\text{char}(R) = 2$ we have $(a - b)^2 = (a - b)(a - b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2 = a^2 + b^2 = a + b = a - b$ so $a - b \in S$.
   (iii) Let $a, b \in S$. Then $a^2 = a$ and $b^2 = b$. Since R is commutative we have $(ab)^2 = abab = a^2b^2 = ab$ so $ab \in S$.
   From (i), (ii) and (iii) S is a subring of R.