Math 312- HW3 Due Date: 31/10/17

1. Find a subring of $\mathbb{Z} \times \mathbb{Z}$ that is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.
Solution:
Consider the set $S = \{(a, a) \mid a \in \mathbb{Z}\}$. This is a subring of $\mathbb{Z} \times \mathbb{Z}$. However it is not an ideal. To see this consider the element $(1, 1) \in S$. We have $(1, 0)(1, 1) = (1, 0) \notin S$. Therefore S is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.

2. In the ring $\mathbb{Z}$, find a positive integer a such that
   (a) $\langle a \rangle = \langle 2 \rangle + \langle 3 \rangle$
   (b) $\langle a \rangle = \langle 3 \rangle + \langle 6 \rangle$
   (c) $\langle a \rangle = \langle 6 \rangle + \langle 8 \rangle$

   Solution:
   (a) 1 (b) 3 (c) 2

3. If I and J are ideals in a commutative ring with unity and $I + J = R$, show that $IJ = I \cap J$.
Solution:
From the definition of the product IJ we see that $IJ \subseteq I \cap J$. For the other containment let $a \in I \cap J$. Since $I + J = R$ and R has unity therefore there exist $s \in I$ and $t \in J$ such that $s + t = 1$. So multiplying this equation by a on the right we get $sa + ta = a$. Since $a \in I \cap J$ and R is commutative therefore $sa + ta = sa + at \in IJ$ so $a \in IJ$. This shows that $I \cap J \subseteq IJ$. Thus $I \cap J = IJ$.

4. Let $R = \mathbb{Z}_8 \times \mathbb{Z}_{30}$. Find all maximal ideals of R, and for each maximal ideal I, identify the size of the field R/I.
Solution:
The maximal ideals of $R = \mathbb{Z}_8 \times \mathbb{Z}_{30}$ are $I_1 = \langle 1 \rangle \times \langle 2 \rangle$, $I_2 = \langle 1 \rangle \times \langle 3 \rangle$, $I_3 = \langle 1 \rangle \times \langle 5 \rangle$, $I_4 = \langle 2 \rangle \times \langle 1 \rangle$. We have $|R/I_1| = 2$, $|R/I_2| = 3$, $|R/I_3| = 5$ and $|R/I_4| = 2$

5. In $\mathbb{Z} \times \mathbb{Z}$, let $I = \{(a, 0) \mid a \in \mathbb{Z}\}$. Show that I is a prime ideal but not a maximal ideal.
Solution:
Note that I is proper since for example $(1, 1) \notin I$. Now suppose that we have $b = (b_1, b_2)$, $c = (c_1, c_2)$ with $bc = (b_1c_1, b_2c_2) \in I$. Then $b_2c_2 = 0$. Since $\mathbb{Z}$ is an integral domain therefore $b_2 = 0$ or $c_2 = 0$. This shows that either $b \in I$ or $c \in I$. i.e. I is a prime ideal of $\mathbb{Z} \times \mathbb{Z}$. Now let $J = \{(a, 2b) \mid a, b \in \mathbb{Z}\}$. Then J is an ideal of $\mathbb{Z} \times \mathbb{Z}$ and $I \subsetneq J \subsetneq \mathbb{Z} \times \mathbb{Z}$. Therefore I is not a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$.

6. Consider the set $S = \{(2a, 2b) \mid a, b \in \mathbb{Z}\}$. Prove or disprove that S is an ideal of $\mathbb{Z} \times \mathbb{Z}$.
Solution:
(i) $(0, 0) \in S$. Therefore $S \neq \emptyset$
(ii) Suppose that $x = (2a, 2b), y = (2a', 2b') \in S$ so $a, a', b, b' \in \mathbb{Z}$. Then $x - y = (2a - 2a', 2b - 2b') = (2(a - a'), 2(b - b'))$. So $x - y \in S$
(iii) Suppose that $x = (2a, 2b) \in S$ and $z = (c, d) \in \mathbb{Z} \times \mathbb{Z}$ so $a, b, c, d \in \mathbb{Z}$. We have $xz = (2ac, 2bd)$. So $xz \in S$. Since $\mathbb{Z} \times \mathbb{Z}$ is commutative, also $zx \in S$.
From (i), (ii) and (iii) S is an ideal of $\mathbb{Z} \times \mathbb{Z}$