1. Let $\phi : R \to S$ be a surjective ring homomorphism and suppose that A is an ideal of R. Prove that $\phi(A) = \{\phi(a) \mid a \in A\}$ is an ideal of S.
   Solution:
   (i) Since $0 \in A$ and $\phi(0) = 0$, therefore $0 \in \phi(A)$ so $\phi(A) \neq \emptyset$
   (ii) Suppose that $x, y \in \phi(A)$, then $x = \phi(a)$ and $y = \phi(b)$ for some $a, b \in A$. We have $x - y = \phi(a) - \phi(b) = \phi(a - b)$ so $x - y \in \phi(A)$
   (iii) Suppose that $x \in \phi(A)$ and $s \in S$. Since $x \in \phi(A)$ therefore $x = \phi(a)$ for some $a \in A$. Also since ϕ is surjective $s = \phi(r)$ for some $r \in R$. Then we have $xs = \phi(a)\phi(r) = \phi(ar) \in \phi(A)$. Also $sx = \phi(r)\phi(a) = \phi(ra) \in \phi(A)$.
   From (i), (ii) and (iii) $\phi(A)$ is an ideal of S.

2. Let $\phi : R \to S$ be a ring homomorphism and suppose that B is an ideal of S. Prove that $\phi^{-1}(B) = \{r \in R \mid \phi(r) \in B\}$ is an ideal of R.
   Solution:
   (i) $\phi(0) = 0 \in B$ so $0 \in \phi^{-1}(B)$. Therefore $\phi^{-1}(B) \neq \emptyset$
   (ii) Suppose $x, y \in \phi^{-1}(B)$. Then $\phi(x) = b$ and $\phi(y) = b'$ for some $b, b' \in B$. We have $\phi(x - y) = \phi(x) - \phi(y) = b - b' \in B$ so $x - y \in \phi^{-1}(B)$.
   (iii) Suppose that $x \in \phi^{-1}(B)$ and $r \in R$. Then $\phi(x) = b$ for some $b \in B$. We have $\phi(rx) = \phi(r)\phi(x) = \phi(r)b$ and $\phi(r)b \in B$ since B is an ideal of S. Therefore $rx \in \phi^{-1}(B)$. Similarly we have $\phi(xr) = \phi(x)\phi(r) = b\phi(r) \in B$ so $xr \in \phi^{-1}(B)$.
   From (i), (ii) and (iii) $\phi^{-1}(B)$ is an ideal of R.

3. Let R and S be nonzero rings with unity and denote their respective unities by 1_R and 1_S. Let $\phi : R \to S$ be a nonzero homomorphism of rings. Prove that if $\phi(1_R) \neq 1_S$ then $\phi(1_R)$ is a zero divisor in S. Deduce that if S is an integral domain then every nonzero ring homomorphism from R to S sends the unity of R to the unity of S.
   Solution:
   First we note that if $\phi(1_R) = 0$ then ϕ is the zero map. To see this note that if $\phi(1_R) = 0$ then $1_R \in \ker\phi$. Since $\ker\phi$ is an ideal of R and $1_R \in \ker\phi$ therefore $\ker\phi = R$ so ϕ is the zero map. As we have assumed ϕ to be a nonzero homomorphism therefore $\phi(1_R) \neq 0$. Now assume that $\phi(1_R) \neq 1_S$. Then we have $\phi(1_R) = \phi(1_R 1_R) = \phi(1_R)\phi(1_R)$ so $\phi(1_R) - \phi(1_R)\phi(1_R) = 0$. This implies that $\phi(1_R)(1_S - \phi(1_R)) = 0$. Similarly we can show that $(1_S - \phi(1_R))\phi(1_R) = 0$. But $\phi(1_R) \neq 0$ and $1_S - \phi(1_R) \neq 0$ (because $\phi(1_R) \neq 1_S$). Therefore $\phi(1_R)$ is a zero divisor of S. If ϕ is a nonzero homomorphism and S is an integral domain then as S has no zero divisors therefore we must have $\phi(1_R) = 1_S$

4. Let I be an ideal of a ring R and S a subring of R. Prove that $I \cap S$ is an ideal of S.
   Solution:
   (i) Since $0 \in I$ and $0 \in S$ therefore $0 \in I \cap S$ so $I \cap S \neq \emptyset$
   (ii) Let $x, y \in I \cap S$. Then $x, y \in I$ and since I is an ideal of R we have $x - y \in I$. Also $x, y \in S$ and since S is a subring of R we have $x - y \in S$. Therefore $x - y \in I \cap S$.
   (iii) Let $x \in I \cap S$ and $s \in S$. Since $x \in I \cap S$ therefore $x \in I$ so $xs \in I$ since I is an ideal of R. Also since $x \in I \cap S$ therefore $x \in S$ so $xs \in S$ since S is a subring of R. Therefore $xs \in I \cap S$. Similarly we can show that $sx \in I \cap S$.
   From (i), (ii) and (iii) $I \cap S$ is an ideal of S.

5. Assume that R is a nonzero commutative ring with unity. Prove that if P is a prime ideal of R and P contains no zero divisors then R is an integral domain.
   Solution:
   We need to show that R has no zero divisors. Assume that $r \in R$ is a zero divisor. Then $rs = 0$ for some nonzero $s \in R$. Note that s is also a zero divisor. Since $rs = 0 \in P$ and P

is a prime ideal of R therefore $r \in P$ or $s \in P$. This is a contradiction since P was assumed not to contain any zero divisors. Therefore R has no zero divisors and so it is an integral domain.

6. Assume that R is a commutative ring. Let I and J be ideals of R and assume that P is a prime ideal of R that contains IJ. Prove that either I or J is contained in P.
   Solution:
   Assume that I is not contained in P. Then there exists $a \in I$ that is not contained in P. Now let $b \in J$. Then $ab \in IJ$ and as IJ is contained in P therefore $ab \in P$. Since P is a prime ideal of R this implies that either $a \in P$ or $b \in P$. Since $a \notin P$ therefore we must have $b \in P$. Since b was an arbitrary element of J this shows that $J \subseteq P$.