1. Let R and S be commutative rings with unity. If ϕ is a surjective homomorphism from R to S and the characteristic of R is nonzero, prove that the characteristic of S divides the characteristic of R.
   Solution:
   Let 1_R and 1_S the respective unities of R and S. Since ϕ is surjective, therefore $\phi(1_R) = 1_S$. Let n be the characteristic of R, then $n \cdot 1_R = 0$ so we have $n \cdot 1_S = n \cdot \phi(1_R) = \phi(n \cdot 1_R) = \phi(0) = 0$. This implies that the additive order of 1_S divides n so the characteristic of S divides the characteristic of R.

2. Prove that the rings R_1 and R_2 are not isomorphic
   (a) $R_1 = \mathbb{Q}, R_2 = \mathbb{Z}$ (b) $R_1 = \mathbb{Z}, R_2 = 2\mathbb{Z}$ (c) $R_1 = M_{22}(\mathbb{Z}_4), R_2 = M_{22}(\mathbb{Z}_6)$
   (d) $R_1 = \mathbb{Z}, R_2 = \mathbb{Z} \times \mathbb{Z}$
   Solution:
   (a) $\mathbb{Q}$ is a field and $\mathbb{Z}$ is not a field so they are not isomorphic
   (b) $\mathbb{Z}$ has unity but $2\mathbb{Z}$ does not have unity so they are not isomorphic
   (c) $\text{char}(M_{22}(\mathbb{Z}_4)) = 4$ whereas $\text{char}(M_{22}(\mathbb{Z}_6)) = 6$ so they are not isomorphic
   (d) $\mathbb{Z}$ is an integral domain whereas $\mathbb{Z} \times \mathbb{Z}$ is not an integral domain (e.g. $(1, 0)(0, 1) = (0, 0)$) so they are not isomorphic

3. Let R and S be nonzero commutative rings with unity and let $\phi : R \to S$ be a surjective ring homomorphism. Prove that S is an integral domain if and only if $\ker \phi$ is a prime ideal of R. Also prove that S is a field if and only if $\ker \phi$ is a maximal ideal of R.
   Solution:
   From the first isomorphism theorem $R/\ker \phi$ is isomorphic to S. Therefore S is an integral domain iff $R/\ker \phi$ is an integral domain iff $\ker \phi$ is a prime ideal of R. Also S is a field iff $S/\ker \phi$ is a field iff $\ker \phi$ is a maximal ideal of S.

4. Let R and S be rings. Consider the map $\phi : R \times S \to R$ given by $\phi((a, b)) = a$.
   (a) Prove ϕ is a ring homomorphism.
   (b) Prove that ϕ is surjective
   (c) Find $\ker \phi$

   Solution:
   (a) Let $(a, b), (a', b') \in R \times S$. We have $\phi((a, b) + (a', b')) = \phi((a + a', b + b')) = a + a' = \phi((a, b)) + \phi((a', b'))$. Also $\phi((a, b)(a', b')) = \phi((aa', bb')) = aa' = \phi((a, b))\phi((a', b'))$. This proves that ϕ is a ring homomorphism.
   (b) Let $a \in R$. Then $\phi((a, 0)) = a$. This proves that ϕ is surjective.
   (c) We have
   $\ker \phi = \{(a, b) \in R \times S \mid \phi((a, b)) = 0\} = \{(a, b) \in R \times S \mid a = 0\} = \{(0, b) \mid b \in S\}$

5. Assume that R is a nonzero ring with unity and S is a nonzero ring. If $\phi : R \to S$ is a surjective ring homomorphism, prove that $\phi(u)$ is a unit in S for every unit u in R.
   Solution:
   Let 1_R be the unit of R. Since ϕ is surjective $\phi(1_R) = 1_S$ where 1_S is the unity of S. Let u be a unit in R. Then $uu^{-1} = 1_R$ and $u^{-1}u = 1_R$. Therefore $1_S = \phi(1_R) = \phi(uu^{-1}) = \phi(u)\phi(u^{-1})$ and similarly $1_S = \phi(u^{-1})\phi(u)$. Therefore $\phi(u)$ is a unit in S.

6. Assume that $\phi : R \to S$ is a ring homomorphism. Assume that I is an ideal of R and J an ideal of S such that $\phi(I) \subseteq J$. Prove that the map $\psi : R/I \to S/J$ given by $\psi(a+I) = \phi(a) + J$ is a ring homomorphism. (Note: You must first show that ψ is well-defined)
Solution:
First we show that ψ is well-defined: Assume that $a+I$ and $b+I$ are elements of R/I with $a+I = b+I$. We must show that $\psi(a+I) = \psi(b+I)$. Since $a+I = b+I$ therefore $a-b \in I$ so $a = b+\alpha$ where $\alpha \in I$. Then we have $\psi(a+I) = \phi(a)+J = \phi(b+\alpha)+J = \phi(b)+\phi(\alpha)+J$. Since $\alpha \in I$ and $\phi(I) \subseteq J$ therefore $\phi(\alpha) \in J$. Therefore $\phi(b)+\phi(\alpha)+J = \phi(b)+J = \psi(b+I)$. This proves that $\psi(a+I) = \psi(b+I)$ so ψ is well-defined.

We now show that ψ is a ring homomorphism. Let $a+I, b+I \in R/I$. Then $\psi((a+I)+(b+I)) = \psi(a+b+I) = \phi(a+b)+J = \phi(a)+\phi(b)+J = (\phi(a)+J)+(\phi(b)+J) = \psi(a+I)+\psi(b+I)$
Also $\psi((a+I)(b+I)) = \psi(ab+I) = \phi(ab)+J = \phi(a)\phi(b)+J = (\phi(a)+J)(\phi(b)+J) = \psi(a+I)\psi(b+I)$ Therefore ψ is a ring homomorphism