1. Determine whether the element is irreducible in the given integral domain
   (a) 5 in $\mathbb{Z}$ (b) 12 in $\mathbb{Z}$ (c) $3x+9$ in $\mathbb{Z}[x]$ (d) $3x+9$ in $\mathbb{Q}[x]$
   Solution:
   (a) irreducible (b) reducible (c) reducible (d) irreducible

2. Let R be a ring and I an ideal of R. Let $I[x]$ be the ideal of $R[x]$ consisting of all polynomials with coefficients in I. Prove that $R[x]/I[x]$ is ring-isomorphic to $(R/I)[x]$
   Solution:
   Consider the map $\phi : R[x] \to (R/I)[x]$ given by $\phi(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n (a_i + I)x^i$
   We now show that ϕ is a ring homomorphism. Let $p(x), q(x) \in R[x]$. Then we may write $p(x) = \sum_{i=0}^n a_i x^i$ and $q(x) = \sum_{i=0}^n b_i x^i$. Note that $p(x)$ and $q(x)$ do not necessarily have degree n.
   Now we have $\phi(p(x)+q(x)) = \phi(\sum_{i=0}^n (a_i+b_i)x^i) = \sum_{i=0}^n ((a_i+b_i)+I)x^i = \sum_{i=0}^n ((a_i+I)+(b_i+I))x^i = \sum_{i=0}^n (a_i+I)x^i + \sum_{i=0}^n (b_i+I)x^i = \phi(p(x)) + \phi(q(x))$
   Also we have $\phi(p(x)q(x)) = \phi(\sum_{k=0}^{2n}(\sum_{i=0}^k a_i b_{k-i})x^k) = \sum_{k=0}^{2n}((\sum_{i=0}^k a_i b_{k-i}) + I)x^k = \sum_{k=0}^{2n}(\sum_{i=0}^k (a_i b_{k-i} + I))x^k = \sum_{k=0}^{2n}(\sum_{i=0}^k (a_i+I)(b_{k-i}+I))x^k = \phi(p(x))\phi(q(x))$
   Therefore ϕ is a ring homomorphism. Now assume that $f(x) = \sum_{i=0}^n (a_i+I)x^i \in (R/I)[x]$. Then $\phi(\sum_{i=0}^n a_i x^i) = f(x)$. Therefore ϕ is surjective.
   $$\begin{aligned}\ker\phi &= \{\sum_{i=0}^n a_i x^i \in R[x] \mid \phi(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n (0+I)x^i\} \\ &= \{\sum_{i=0}^n a_i x^i \in R[x] \mid \sum_{i=0}^n (a_i+I)x^i = \sum_{i=0}^n (0+I)x^i\} \\ &= \{\sum_{i=0}^n a_i x^i \in R[x] \mid a_i + I = 0 + I \text{ for all } 0 \le i \le n\} \\ &= \{\sum_{i=0}^n a_i x^i \in R[x] \mid a_i \in I \text{ for all } 0 \le i \le n\} = I[x]\end{aligned}$$
   Therefore by the first isomorphism theorem $R[x]/I[x]$ is isomorphic to $(R/I)[x]$

3. Use the previous problem to prove that the set of all polynomials with even integer coefficients is a prime ideal of $\mathbb{Z}[x]$
   Solution:
   The set of all polynomials with even integer coefficients in $2\mathbb{Z}[x]$. By the previous problem $\mathbb{Z}[x]/2\mathbb{Z}[x]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})[x]$. Since $\mathbb{Z}/2\mathbb{Z}$ is a an integral domain therefore $(\mathbb{Z}/2\mathbb{Z})[x]$ is also an integral domain so by the isomorphism $\mathbb{Z}[x]/2\mathbb{Z}[x]$ is an integral domain. It follows that $2\mathbb{Z}[x]$ is a prime ideal of $\mathbb{Z}[x]$.

4. Find a polynomial of degree greater than zero in $\mathbb{Z}_4[x]$ that is a unit. Prove your result.
   Solution:
   Consider the polynomial $p(x) = 2x+1 \in \mathbb{Z}_4[x]$. We have
   $(2x+1)(2x+1) = 4x^2 + 4x + 1 = 1$. Therefore $p(x)$ is a unit.

5. Suppose that a and b belong to an integral domain, $b \neq 0$ and a is not a unit. Show that $\langle ab \rangle$ is a proper subset of $\langle b \rangle$.
   Solution:
   Suppose that $\langle ab \rangle = \langle b \rangle$. Then $b = abr = arb$ for some $r \in R$. Since R is an integral domain and $b \neq 0$ therefore $ar = 1$. Since R is commutative we have $ra = 1$ also so a is a unit. This is contradiction which proves that $\langle ab \rangle$ is indeed a proper subset of $\langle b \rangle$.

6. Let $R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{Z} \right\}$. Prove that R is ring-isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Solution:

Let $\phi : R \to \mathbb{Z} \times \mathbb{Z}$ be defined as $\phi\Big(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Big) = (a, b)$.

Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}$.

We have $\phi(A+B) = \phi\Big(\begin{bmatrix} a+a' & 0 \\ 0 & b+b' \end{bmatrix} \Big) = (a+a', b+b') = (a,b) + (a',b') = \phi(A) + \phi(B)$

Also $\phi(AB) = \phi\Big(\begin{bmatrix} aa' & 0 \\ 0 & bb' \end{bmatrix} \Big) = (aa', bb') = (a,b)(a',b') = \phi(A)\phi(B)$

Therefore ϕ is a ring homomorphism. Now let $(a,b) \in \mathbb{Z} \times \mathbb{Z}$. We have $\phi\Big(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Big) = (a, b)$ so ϕ is surjective.

$\ker \phi = \Big\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid \phi\Big(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Big) = (0,0) \Big\} = \Big\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid (a,b) = (0,0) \Big\} = \Big\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Big\}$.

Therefore ϕ is injective.

This shows that R is ring-isomorphic to $\mathbb{Z} \times \mathbb{Z}$.