All rings in the test are nonzero

Question 1 (6+4 points)

(a) Let I be an ideal in a ring R. Define $[R : I] = \{r \in R \mid xr \in I \text{ for every } x \in R\}$. Prove that $[R : I]$ is an ideal of R that contains I.
Solution:
(i) for any $x \in R$ we have $x0 = 0 \in I$ so $0 \in [R : I]$. Therefore $[R : I] \neq \emptyset$
(ii) Suppose that $r, s \in [R : I]$. For any $x \in R$ we have $xr \in I$ and $xs \in I$ therefore $x(r - s) = xr - xs \in I$ since I is an ideal of R. This shows that $r - s \in [R : I]$
(iii) Suppose that $r \in [R : I]$ and $s \in R$. We must show that $rs \in [R : I]$ and $sr \in [R : I]$. Let $x \in R$. We have $x(rs) = (xr)s$. Since $r \in [R : I]$ therefore $xr \in I$ so $(xr)s \in I$ since I is an ideal of R. This shows that $x(rs) \in I$ so $rs \in [R : I]$. Also we have $x(sr) = (xs)r \in I$ (because $r \in [R : I]$) so $sr \in [R : I]$
From (i), (ii) and (iii) $[R : I]$ is an ideal of R
Now let $a \in I$ and $x \in R$. Then $xa \in I$ since I is an ideal of R. This proves that $a \in [R : I]$ so $I \subseteq [R : I]$

(b) Suppose that R is a commutative ring and P is a prime ideal of R. Prove that $[R : P] = P$.
Solution:
From part (a) we have $P \subseteq [R : P]$. We need to show that $[R : P] \subseteq P$. Assume that $r \in [R : P]$. Then by the definition of $[R : P]$ we have $rr \in P$. Since P is a prime ideal of R this implies that $r \in P$. Therefore $[R : P] \subseteq P$ thus proving that $[R : P] = P$.

Question 2 (6+4 points)

(a) An element a of a ring R is called nilpotent if $a^n = 0$ for some integer $n \geq 1$. Now let R be a commutative ring and let N be the set all the nilpotent elements of R. Prove that N is an ideal of R.
Solution:
(i) We have $0^1 = 0$ so $0 \in N$. Therefore $N \neq \emptyset$
(ii) Suppose that $a, b \in N$. Then $a^{m_1} = 0$ and $b^{m_2} = 0$ for some positive integers m_1 and m_2. Note that if t is an integer with $t > m_1$ then $t = m_1 + s$ for some positive integer s so $a^t = a^s a^{m_1} = 0$. A similar observation applies to b and m_2. Let $m = \max\{m_1, m_2\}$. Note that we have $a^m = 0$ and $b^m = 0$. We also have $(-b)^m = (-1)^m b^m = 0$. Now let $n = 2m$. By the binomial theorem we have $(a - b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i (-b)^{n-i}$. Consider a typical term in this sum $\binom{n}{i} a^i (-b)^{n-i}$. We cannot have $i < m$ and $n - i < m$ because that would imply that $n = i + n - i < 2m$. Therefore either $a^i = 0$ or $(-b)^{n-i} = 0$. This shows that $(a - b)^n = 0$. Thus $a - b \in N$
(iii) Suppose that $a \in N$ and $r \in R$. Since a is nilpotent $a^n = 0$ for some positive integer n. Then we have $(ra)^n = r^n a^n = 0$. Since R is commutative we also have $(ar)^n = 0$. Thus $ra, ar \in N$
From (i), (ii) and (iii) N is an ideal of R.

(b) Let R and N be as in part (a). Prove that R/N is a ring with no nonzero nilpotent elements.
Solution:
Suppose that $a + N$ is nilpotent. Then $(a + N)^n = 0 + N$ for some positive integer n. Then $a^n + N = (a + N)^n = 0 + N$ so $a^n \in N$. Therefore $a^n \in N$ which implies that $(a^n)^m = 0$ for some positive integer m so $a^{nm} = 0$. This implies that $a \in N$ so $a + N = 0 + N$

Question 3 (4+3+3 points)

(a) Let R be a finite commutative ring with unity. Prove that every prime ideal of R is maximal (Hint: If P is a prime ideal of R consider the quotient R/P).
Solution:
Suppose that P is a prime ideal of R. Then R/P is an integral domain. Since R is finite therefore R/P is a finite integral domain so is a field. This implies that P is a maximal ideal of R.

(b) Suppose that F is a field and I is an ideal of F. Prove that $I = \{0\}$ or $I = F$.
Solution:
Suppose that $I \neq \{0\}$. Then I contains a nonzero element a of F. Since F is a field and a is not zero therefore a is a unit. Then since I is an ideal we have $1 = a^{-1}a \in I$. Now let $c \in F$ then $c = c1 \in I$. Therefore $I = F$

(c) Let R be a ring with additive identity 0 and $a \in R$. Prove that $0a = 0$.
Solution:
We have $0a = (0 + 0)a = 0a + 0a$. Adding $-0a$ to both sides of this equation we get $0 = 0a$

Question 4 (10 points)

Label each of the following statements as True **(T)** or False **(F)**

(i) $\langle 1 \rangle \times \langle 0 \rangle$ is a maximal ideal of $\mathbb{Z}_4 \times \mathbb{Z}_2$ (**T**)

(ii) If R is an integral domain and G is a finite group, then the group ring RG is an integral domain (**F**)

(iii) If R is a ring and $a \in R$ is a unit, then a is not a zero divisor (**T**)

(iv) If R is a ring with unity 1 and $5 \cdot 1 = 0$ in R, then $\text{char}(R) = 5$ (**T**)

(v) There exists an integral domain of characteristic 6 (**F**)