Question 1 (6+6+4+2 points)
Suppose that R and S are rings and let $\phi : R \to S$ be a surjective ring homomorphism. Let J be an ideal of S and $I = \phi^{-1}(J) = \{r \in R \mid \phi(r) \in J\}$
(a) Prove that I is an ideal of R.
Solution:
(i) $\phi(0) = 0 \in J$ so $0 \in I$. Therefore $I \neq \emptyset$
(ii) Suppose $x, y \in I$. Then $\phi(x) = b$ and $\phi(y) = b'$ for some $b, b' \in J$. We have $\phi(x - y) = \phi(x) - \phi(y) = b - b' \in J$ so $x - y \in I$.
(iii) Suppose that $x \in I$ and $r \in R$. Then $\phi(x) = b$ for some $b \in J$. We have $\phi(rx) = \phi(r)\phi(x) = \phi(r)b$ and $\phi(r)b \in J$ since J is an ideal of S. Therefore $rx \in I$. Similarly we have $\phi(xr) = \phi(x)\phi(r) = b\phi(r) \in J$ so $xr \in I$.
From (i), (ii) and (iii) I is an ideal of R.

(b) Consider the map $\psi : R/I \to S/J$ given by $\psi(a + I) = \phi(a) + J$. Prove that ψ is a ring homomorphism (Note: You must first prove that ψ is well-defined).
Solution:
First we show that ψ is well-defined: Assume that $a + I$ and $b + I$ are elements of R/I with $a + I = b + I$. We must show that $\psi(a + I) = \psi(b + I)$. Since $a + I = b + I$ therefore $a - b \in I$ so $a = b + \alpha$ where $\alpha \in I$. Then we have $\psi(a + I) = \phi(a) + J = \phi(b + \alpha) + J = \phi(b) + \phi(\alpha) + J$. Since $\alpha \in I$ and $\phi(I) \subseteq J$, therefore $\phi(\alpha) \in J$. Therefore $\phi(b) + \phi(\alpha) + J = \phi(b) + J = \psi(b + I)$. This proves that $\psi(a + I) = \psi(b + I)$ so ψ is well-defined.
We now show that ψ is a ring homomorphism. Let $a + I, b + I \in R/I$. Then
$\psi((a + I) + (b + I)) = \psi(a + b + I) = \phi(a + b) + J = \phi(a) + \phi(b) + J =$
$(\phi(a) + J) + (\phi(b) + J) = \psi(a + I) + \psi(b + I)$
Also
$\psi((a+I)(b+I)) = \psi(ab+I) = \phi(ab)+J = \phi(a)\phi(b)+J = (\phi(a)+J)(\phi(b)+J) = \psi(a+I)\psi(b+I)$
Therefore ψ is a ring homomorphism

(c) Prove that ψ is injective
Solution:
Suppose that $a + I \in \ker \psi$. Then $\phi(a) + J = \psi(a + I) = 0 + J$. Therefore $\phi(a) \in J$ which implies that $a \in I$ so $a + I = 0 + I$. This proves that $\ker \psi = \{0 + I\}$ so ψ is injective.

(d) Prove that ψ is surjective
Solution:
Let $b + J \in S/J$ so $b \in S$. Since ϕ is surjective therefore there exists $a \in R$ such that $\phi(a) = b$. Then $\psi(a + I) = \phi(a) + J = b + J$ so ψ is surjective.

Question 2 (5 points)
Let F be a field and R a ring. Let $\phi : F \to R$ be a ring homomorphism. Prove that ϕ is either injective or the zero map (Hint: Consider the different possibilities for $\ker \phi$)
Solution:
$\ker \phi$ is an ideal of F. Since F is a field, the only possibilities for $\ker \phi$ are $\ker \phi = \{0\}$ and $\ker \phi = F$. If $\ker \phi = \{0\}$ then ϕ is injective whereas if $\ker \phi = F$ then ϕ is the zero map.

Question 3 (4+2+3+2+2 points)

Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{Z} \right\}$. Let $\phi : R \to \mathbb{Z}$ be the map defined by $\phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = a - b$

(a) Prove that ϕ is a ring homomorphism
Solution:
Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $B = \begin{bmatrix} a' & b' \\ b' & a' \end{bmatrix}$.
We have
$\phi(A+B) = \phi\left(\begin{bmatrix} a+a' & b+b' \\ b+b' & a+a' \end{bmatrix} \right) = (a+a') - (b+b') = (a-b) + (a'-b') = \phi(A) + \phi(B)$ Also
$\phi(AB) = \phi\left(\begin{bmatrix} aa'+bb' & ab'+ba' \\ ba'+ab' & bb'+aa' \end{bmatrix} \right) = (aa'+bb') - (ab'+ba') = aa' - ab' - ba' + bb' =$
$(a-b)(a'-b') = \phi(A)\phi(B)$
This proves that ϕ is a ring homomorphism

(b) Find $\ker \phi$
Solution:

$$\begin{aligned} \ker \phi &= \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \;\middle|\; \phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \;\middle|\; a - b = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \;\middle|\; a = b \right\} \\ &= \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \;\middle|\; a \in \mathbb{Z} \right\} \end{aligned}$$

(c) Show that $R/\ker \phi$ is isomorphic to $\mathbb{Z}$
Solution:
Let $a \in \mathbb{Z}$. Then $\phi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right) = a$. This proves that ϕ is surjective. Therefore by the first isomorphism theorem we have $R/\ker \phi$ is isomorphic to $\mathbb{Z}$.

(d) Is $\ker \phi$ a prime ideal? explain
Solution:
$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $B = \begin{bmatrix} a' & b' \\ b' & a' \end{bmatrix}$ We have $AB = \begin{bmatrix} aa'+bb' & ab'+ba' \\ ba'+ab' & bb'+aa' \end{bmatrix} = BA$ so R is commutative.
From part (c) we have that $R/\ker \phi$ is isomorphic to $\mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain therefore $R/\ker \phi$ is an integral domain. Since R is commutative, this implies that $\ker \phi$ is a prime ideal of R.

(e) Is $\ker \phi$ a maximal ideal? explain
Solution:
From part (c) we have that $R/\ker \phi$ is isomorphic to $\mathbb{Z}$. Since $\mathbb{Z}$ is not a field therefore $R/\ker \phi$ is not a field. Since R is commutative (shown in part (d)), this implies that $\ker \phi$ is not a maximal ideal of R.

Question 4 (4 points)

Use the Euclidean algorithm to find a greatest common divisor of 3084 and 1424 in $\mathbb{Z}$

Solution:

3084=2(1424)+236

1424=6(236)+8

236=29(8)+4

8=2(4)+0

Therefore 4 is a greatest common divisor of 3084 and 1424

Question 4 (10 points)

Label each of the following statements as True **(T)** or False **(F)**

(i) If $\phi : R \to S$ is a ring homomorphism and I is an ideal of R
then $\phi(I)$ is an ideal of S (**F**)

(ii) If $\phi : R \to S$ is a ring homomorphism and $\phi(u) = 0$ for some unit u of R
then ϕ is the zero map (**T**)

(iii) $\langle 0 \rangle$ is a prime ideal of $\mathbb{Z}$ (**T**)

(iv) Every field is a Euclidean domain (**T**)

(v) $M_{22}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ (**F**)